Algebras of the Giry monad

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Overview

1. Giry monad
2. Examples of algebras of probability monads
3. Topologizing Giry algebras
4. Ordering Giry algebras
5. Giry algebra-valued random variables

Giry monad

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  For a measurable map $f : \Omega_1 \to \Omega_2$, **pushing forward** along f defines a measurable map $Gf : G\Omega_1 \to G\Omega_2$.

  This gives an **endofunctor** G : **Mble** $\to$ **Mble**.

- There is a measurable map $\mu_\Omega : G G\Omega \to G\Omega$:

$$\mu_\Omega(\mathbf{P})(A) := \int_{\lambda \in G\Omega} \lambda(A)\mathbf{P}(\mathrm{d}\lambda),$$

for all $\mathbf{P} \in GG\Omega$ and measurable subsets $A \subseteq \Omega$.

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- We have a map $\eta_\Omega : \Omega \to G\Omega$:

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- We have a map $\eta_\Omega : \Omega \to G\Omega$:

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  for all $\omega \in \Omega$.
- These form natural transformation $\mu : GG \to G$ and $\eta : 1_{\mathbf{Mble}} \to G$ and (G, μ, η) forms a monad, *the Giry monad*.

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  We have that

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  Therefore, $([0,1], \int)$ is a Giry algebra.

Examples of Giry algebras

- For the set $\{0, \infty\}$ we can define a map $\alpha : G(\{0, \infty\}) \to \{0, \infty\}$ by

$$\alpha\left(\lambda\delta_0 + \overline{\lambda}\delta_\infty\right) := \begin{cases} 0 & \text{if} \quad \lambda = 1, \\ \infty & \text{otherwise.} \end{cases}$$

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  For a probability measure $\mathbf{P}$ on $G(\{0, \infty\})$, we have that

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Other examples: $[0, 1)$, $[0, \infty]$, $[0, 1]^{\mathbb{R}}$, lattices, ...

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Examples: $\mathbb{R}$, vector spaces, lattices, ...

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Given a convex compact Hausdorff space K, a structure map $R(K) \to K$ can be defined by sending every probability measure to its *barycenter*.

Topologizing Giry algbras

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Define the subspace

$$W_1 := \left\{ \sum_{n=1}^{N} \lambda_n (p_n - \delta_{\gamma(p_n)}) \mid \lambda_n \in \mathbb{R}, p_n \in D(C) \text{ for all } n \right\}$$

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This defines a left adjoint functor $\mathbf{Mble}^D \to \mathbf{Vect}$ to the forgetful functor $\mathbf{Vect} \to \mathbf{Mble}^D$.

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$$M_f DC \underset{M\gamma}{\overset{\mu}{\rightrightarrows}} MC \longrightarrow V_1(C)$$

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Indeed,

$$\sum_{n=1}^{N} \lambda_n (p_n - \delta_{\gamma(p_n)}) = \mu(\mathbf{Q}) - \mathbf{Q} \circ \gamma^{-1},$$

where $\mathbf{Q} = \sum_{n=1}^{N} \lambda_n \delta_{p_n}$ in $M_f DC$.

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$$\phi_1 : C \to V_1(C) : c \mapsto [\delta_c].$$

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In this case we say that (C, γ) satisfies the (first) cancellation property (C1).

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i.e $V_2(A) = MA/W_2$, where

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However, this LCTVS might not be *Hausdorff*.

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$$\phi_{2,3} : V_2(A) \to V_3(A)$$

be the corresponding quotient map.

Associated Hausdorff LCTVS

Proposition

Let (A, α) be a Giry algebra, then $V_3(A) = MA/W_3$, where

$$W_3 = \{\mu \mid \mu(f) = 0 \quad \text{for all algebra morphisms } f : A \to [0,1]\}.$$

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The Hausdorff LCTVS structure on $V_3(A)$ is again given by the family of seminorms,

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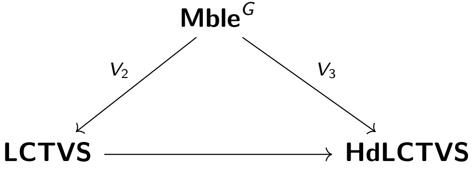
This gives a commutive diagram:



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- $\phi_{2,3}$ is an isomorphism and (A, α) satifies (C1).

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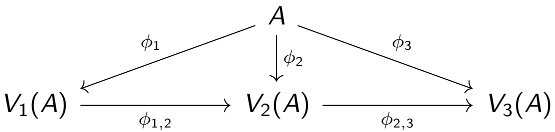
In this case we say that (A, α) satisfies the second cancellation property (C2).

Associated vector spaces

The vector spaces are linked to each other in the following way:



Cancellation properties

We have inclusion functors:

$$\mathbf{Mble}^G_{C2} \longrightarrow \mathbf{Mble}^G_{C1} \longrightarrow \mathbf{Mble}^G$$

Cancellation properties

These have **left adjoints**:

$$\mathbf{Mble}^G_{C2} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathbf{Mble}^G_{C1} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathbf{Mble}^G$$

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The units of the adjunctions are given by ϕ_1 and ϕ_3.

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Theorem

Let (A, α) be a Giry algebra, then $\phi_3(A) \subseteq V_3(A)$ is a **convex, relatively compact, Hausdorff** subspace.

Topologizing Giry algebras

Proof: The vector space $M(A)$ of finite signed measures becomes topological, by endowing it with the topology generated by the maps

$$\rho_f : M(A) \to \mathbb{R} : \mu \mapsto |\mu(f)|$$

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By the Banach-Alaoglu theorem, it follows that the subset

$$\left\{ \mu \in M(A) \mid \sup_{\|f\|_\infty < 1} |\mu(f)| \leq 1 \right\}$$

is a compact subset of $M(A)$.

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There is a continuous linear map $p : M(A) \to V_3(A)$, hence $p(U)$ is a compact subset of V_A.

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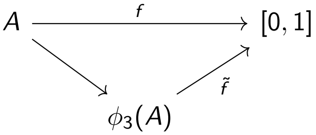
There is a continuous linear map $p : M(A) \to V_3(A)$, hence $p(U)$ is a compact subset of V_A.

Furthermore, we have that $\phi_3(A) \subseteq p(U)$ and therefore it is relatively compact.

Topologizing Giry algebras

Every algebra morphism $f : A \to [0,1]$ factors uniquely through $A \to \phi_3(A)$.



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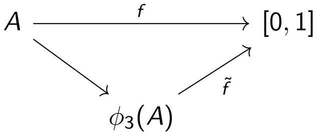

Proposition

A map $f : A \to [0,1]$ is an algebra morphism if and only if $\tilde{f} : \phi_3(A) \to [0,1]$ is convex and continuous.

Topologizing Giry algebras

Proof: Let $\mathbb{P}$ be a probability measure on A. There exist a net $(p_i)_i$ such that

$$p_i(g) \to \mathbb{P}(g)$$

for all convex morphisms $g : A \to [0,1]$.

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$$\mathbb{P}(f) = \lim_i p_i(f) = \lim_i f(\alpha(p_i)) = \tilde{f}(\lim_i \phi_3(\alpha(p_i)) = \tilde{f}\phi_3\alpha(\mathbb{P}) = f(\alpha(\mathbb{P})).$$

Topologizing Giry algebras

Corollary

Let (A, α) be a Giry algebra such that $V_3(A)$ is *finite*-dimensional.

- (A, α) satisfies (C1) if and only if (A, α) satisfies (C2).
- Every convex map $f : A \to [0, 1]$ is an algebra morhpsism.

Topologizing Giry algebras

Theorem

Let (A, α) be a Giry algebra. For a probability measure $\mathbb{P}$ on A,

$$\phi_3(\alpha(\mathbb{P})) = \int \phi_3 \mathrm{d}\mathbb{P}.$$

[1] Note that it is bounded because $\overline{\phi_3(A)}$ is compact.

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Here we used the *Pettis integral* of $\phi_3 : A \to V_3(A)$.

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Proof: Because $\phi_3(A)$ is a convex, relatively compact subset of $V_3(A)$, we know that the Pettis integral exists. Let $g : V_3(A) \to \mathbb{R}$ be a linear continuous functional.

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Topologizing Giry algebras

Theorem

Let (A, α) be a Giry algebra. For a probability measure $\mathbb{P}$ on A,

$$\phi_3(\alpha(\mathbb{P})) = \int \phi_3 \mathrm{d}\mathbb{P}.$$

Here we used the *Pettis integral* of $\phi_3 : A \to V_3(A)$.

Proof: Because $\phi_3(A)$ is a convex, relatively compact subset of $V_3(A)$, we know that the Pettis integral exists. Let $g : V_3(A) \to \mathbb{R}$ be a linear continuous functional. Then $g \circ \phi_3$ is an algebra morphism[1].

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$$g(\phi_3(\alpha(\mathbb{P}))) = \int g\phi_3 \mathrm{d}\mathbb{P}.$$

By the defining property of the Pettis integral, the statement follows.

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Radon-Giry algebra adjunction

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We obtain a **full and faithful** functor $R : \mathbf{CH}^R \to \mathbf{Mble}^G$.

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The functor $R : \mathbf{CH}^R \to \mathbf{Mble}^G$ is right adjoint to $L : \mathbf{Mble}^G \to \mathbf{CH}^R$.

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Theorem

The functor $R : \mathbf{CH}^R \to \mathbf{Mble}^G$ is right adjoint to $L : \mathbf{Mble}^G \to \mathbf{CH}^R$. Moreover, the counit is a natural isomorphism and $\eta_{(A,\alpha)}$ is monic if and only if (A, α) satisfies (C2).

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Let **RelCompConv** be the category of relatively compact, convex subsets of locally convex topological vector spaces.

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Theorem

The categories $\mathbf{Mble}^G_{C2}$ and **RelCompConv** are equivalent.

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By Lemma 5 in [1], this is equivalent to saying that *there exists* a $\lambda \in (0,1)$ such that $\lambda a + \overline{\lambda} b = b$.

Proposition

$P(A) := (A, \leq)$ forms a partially ordered set and there is a functor $P : \mathbf{Mble}^G \to \mathbf{Pos}$.

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Example: $[0, n] \times \{0\} \leq (\infty, 0)$ in $[0, \infty]^2$ for all $n \geq 0$.

Future work: Infinite elements

Consider the Giry algebra $([0, \infty], \alpha)$, where $\alpha(\mathbb{P}) = \int_0^\infty x\mathbb{P}(\mathrm{d}x)$.

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$$\begin{aligned}\alpha(\mathbb{P}) &= \int_0^\infty x\mathbb{P}(\mathrm{d}x)\\ &= \lim_n \int_0^n x\mathbb{P}(\mathrm{d}x)\\ &= \lim_n \frac{1}{\mathbb{P}([0,n])}\int_0^n x\mathbb{P}(\mathrm{d}x)\\ &= \lim_n \alpha\left(\frac{\mathbb{P}(-\cap[0,n])}{\mathbb{P}([0,n])}\right).\end{aligned}$$

Future work: Infinite elements

Consider the Giry algebra $([0,\infty],\alpha)$, where $\alpha(\mathbb{P}) = \int_0^\infty x\mathbb{P}(\mathrm{d}x)$. This algebra is not (C2). For a probability measure $\mathbb{P}$ on $[0,\infty)$,

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\alpha(\mathbb{P}) &= \int_0^\infty x\mathbb{P}(\mathrm{d}x) \\
&= \lim_n \int_0^n x\mathbb{P}(\mathrm{d}x) \\
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&= \lim_n \alpha\left(\frac{\mathbb{P}(- \cap [0,n])}{\mathbb{P}([0,n])}\right).
\end{aligned}$$

Note that $[0,n]$ is an algebra that does satisfy (C2).

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For an infinite element c, consider the collection

$$S_c := \{B \subseteq A \mid B \leq c \text{ and } B \text{ is a (C2) subalgebra.}\}$$

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Define $B_c := \bigcup S_c$.

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For an infinite element c, consider the collection

$$S_c := \{B \subseteq A \mid B \leq c \text{ and } B \text{ is a (C2) subalgebra.}\}$$

Define $B_c := \bigcup S_c$. For $c_1 \leq c_2$, we have that $B_{c_1} \subseteq B_{c_2}$.

Goal: Can we make sense of the following?

- For a probability measure $\mathbb{P}$ on B_c,

$$\text{"} \quad \alpha(\mathbb{P}) = \lim_{B \in S_c} \int \phi_B \mathrm{d}\mathbb{P} \quad \text{"}.$$

- For a probability measure $\mathbb{P}$ on A,

$$\text{"} \quad \alpha(\mathbb{P}) = \lim_{c \in P(A)} \alpha(\mathbb{P}_{B_c}) \quad \text{"}.$$

Giry algebra-valued random variables

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Note that *every* such random variable is integrable.

Example: For a random variable f taking values in the Giry algebra $([0,1], \int)$, we have

$$\mathbb{E}[f] = \int (\mathbb{P} \circ f^{-1}) = \int x \mathbb{P} \circ f^{-1}(\mathrm{d}x) = \int f \mathrm{d}\mathbb{P}.$$

Conditional expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

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For a random variable f taking values in a Giry algebra (A, α), a **conditional expectation of f with respect to $\mathcal{G}$** is a $\mathcal{G}$-measurable random variable $g : \Omega \to A$ such that

$$\alpha(\mathbb{P}_E \circ f^{-1}) = \alpha(\mathbb{P}_E \circ g^{-1})$$

for all $E \in \mathcal{G}$ such that $\mathbb{P}(E) \neq 0$.

Here $\mathbb{P}_E$ is defined as $\frac{\mathbb{P}(- \cap E)}{\mathbb{P}(E)}$.

Conditional expectation

Proposition

Let (A, α) be a σ-algebra such that $V_3(A)$ satisfies the *Radon-Nikodym property*.

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Let (A, α) be a σ-algebra such that $V_3(A)$ satisfies the *Radon-Nikodym property*. Then conditional expectation of random variables valued in A *exist* and are *almost surely unique*.

References

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